CS522 - Option Pricing: Building the Lattice

Before we describe how price lattices will be built, let us review some of conclusions of the previous lecture. We have argued before that stock price returns follow a normal distribution, while stock prices themselves follow a log-normal distribution. We have also shown (semi)informally that recombining lattices can be used to generate approximations of normal distributions. This motivated our study of our one-period binomial model.

0.1 One-Period Binomial Model

There are only two instruments to invest in at time 0: a stock S and a money-market account (or bond) B. The money-market account has a deterministic continuously compounded return r, yielding a total of Be^{rt} at time t, for an investment of B[1] at time 0. The stock price will have one of two possible random values at time t: S_u (with probability p) or S_d (with probability $1-p$). The state in which S_u occurs will be called the "up" state; the state corresponding to S_d is called the "down" state.

In addition, we consider an arbitrary payoff X at time t. The value of this payoff can, but does not have to, depend on the state of the economy (in fact, the stock price) at time t. This dependency is shown explicitly by denoting the payoff in the "up" state by X_u, and the payoff in the "down" state by X_d.

Our goal is to determine a portfolio of the stock and the money market account so that we can perfectly reproduce the payoff X, irrespective of the state of the world at t. If such a portfolio exists, let us denote by n_S the number of units of stock, also, let n_B denote the number of units of the money market account in the respective portfolio. We will allow for both positive and negative holdings, as well as for fractional units of the stock and the money market account.

The equation below reproduces the payoff of X at time t:

$$\begin{cases} n_s S_u + n_B B e^{rt} = X_u \\ n_s S_d + n_B B e^{rt} = X_d \end{cases}$$

Solving it, we get:

$$\begin{cases} n_s = \frac{X_u - X_d}{S_u - S_d} \\ n_B = \frac{1}{Be^{rt}}(X_u - n_s S_u) \end{cases}$$

The time 0 value of the portfolio that reproduces the payout is

$$V = n_s S + n_B B = n_s S + \frac{1}{e^{rt}}(X_u - n_s S_u)$$

By introducing the quantity q as defined below, we can rewrite the value of the initial portfolio to be:

[1] We assumed implicitly that the money market account is denominated in units of B. If you assume that the money market account is denominated in dollars, you can set $B = 1$.

$$
\begin{aligned}
q &= \frac{Se^{rt} - S_d}{S_u - S_d} = \frac{e^{rt} - \frac{S_d}{S}}{\frac{S_u}{S} - \frac{S_d}{S}} \\
V &= e^{-rt}\left[qX_u + (1-q)\,X_d\right]
\end{aligned}
$$

We have argued last time that we must have

$$0 < q < 1.$$

If this is so, then q can be interpreted as a probability; the value of the portfolio that reproduces X will then be the expectation of the payoff under the probability measure associated with q:

$$V = e^{-rt}\mathbb{E}_q[X] = \mathbb{E}_q[e^{-rt}X].$$

The second equality is justified by the assumption that the interest rate is constant - and known - in the initial state.

If you review our discussion in the preceding lecture, you will note that we have used arbitrage considerations to argue that we can not have $\frac{S_d}{S} > e^{rt}$, or $e^{rt} > \frac{S_u}{S}$. We have not discussed what happens when $e^{rt} = \frac{S_u}{S}$, or when $\frac{S_d}{S} = e^{rt}$.

Let us consider the first situation, when $e^{rt} = \frac{S_u}{S}$. Intuitively, this relationship means that the guaranteed (deterministic) return on the money market account is the same as the return in the "up" state. Since we assumed[2] that $S_u > S_d$, this means that the best return on the stock will equal, but not exceed, the known return on the money market account. The implied strategy is thus quite intuitive: invest in the money market, and finance your investment by selling stocks. This is what we will do: we sell short the stock at time 0, for S. We invest this amount in the money market account until time t, when the money market account will have a value of $Se^{rt} = S_u$. At this time we also buy back the stock at its current price S_d, or S_u. The payoff of the strategy is $Se^{rt} - S_u = 0$, if the "up" state occurs, or $Se^{rt} - S_d = S_u - S_d > 0$, if the "down" state occurs. Thus we never lose money, and we sometimes make money (when the "down" state occurs). The expected amount of earnings is $(1-p)(S_u - S_d) > 0$. This is a more general arbitrage situation than the deterministic situations we have seen before. Still, it is a "money pump," which would make someone infinitely rich if it could be repeated indefinitely. Given our "no arbitrage" assumption, we can thus eliminate the possibility that $e^{rt} = \frac{S_u}{S}$. Since we already knew that $e^{rt} > \frac{S_u}{S}$ is not possible, we must have that $e^{rt} < \frac{S_u}{S}$.

Using similar reasoning we can show that $e^{rt} > \frac{S_d}{S}$. We thus get the strict inequality $\frac{S_d}{S} < e^{rt} < \frac{S_u}{S}$, which, in turn, implies that $0 < q < 1$.

Given these insights, we note that neither the value of the replicating portfolio at time 0, nor the values of n_s and n_B depend on p, the true probability of the "up" state occuring at time t. For our purposes, we can disregard p completely, and focus on q.

[2] Last time we assumed that $S_d \neq S_u$, and that nothing is lost if we assume that $S_u > S_d$. What happens if $S_u = S_d$.

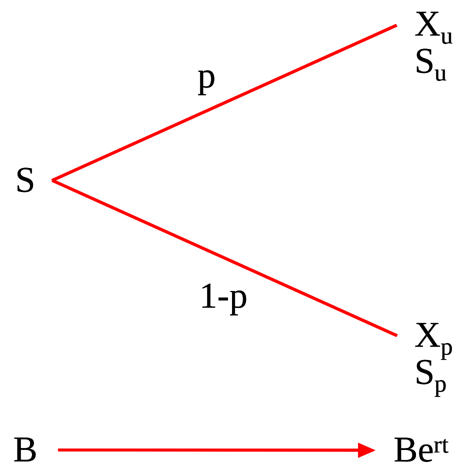

Figure 1: One-period binomial model.

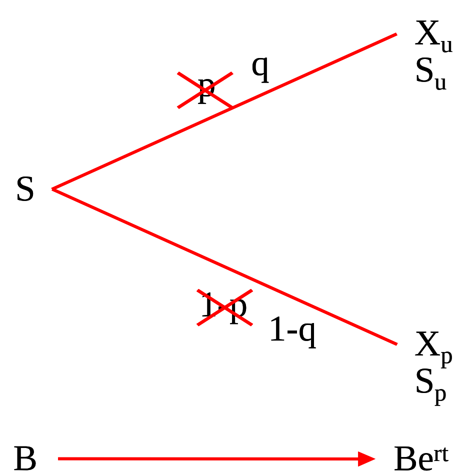

Figure 2: One-period binomial model with the true probabilities replaced by the equivalent [martingale] probabilities.

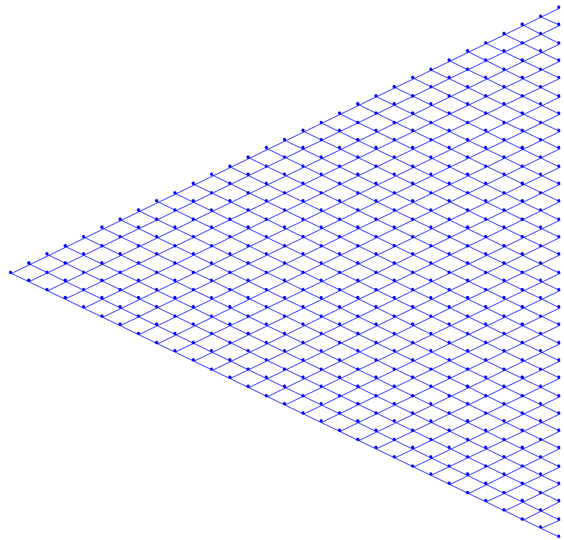

Figure 3: A multi-period lattice. How do we need to change prices at each step so that the nodes correspoding to the same number of "up" and "down" states occuring will recombine?

Under the equivalent probability q the time 0 value of the payoff X - which must equal the value of the replicating portfolio (why?) - is the discounted expected time t payoff of X. Under the true probability p, no such simple dependency exists. It is thus more practical for us to work with q.

0.2 Constructing a Multi-Period Lattice

How can we build a recombining lattice that has the right properties?

Knowing the initial price, we would want to find a systematic way to change prices associated with lattice notes so that recombining nodes will correspond naturally to identical prices.

Remember that we have introduced lattices in order to simulate normal distributions, which are distributions of returns of the stock price, not of the stock price itself. For simplicity, let us assume that the one-period return in all "up" states is always the same , and it is equal to $\log U$; similarly, the one-period return in all "down" states is always the same, and equal to $\log D$. Thus, if a path in the lattice traverses n edges, going "up" k times, and "down" $n-k$ times, the return on the stock price will be $k \log U + (n-k) \log D$, irrespective of the order in which the "up" and "down" states have been encountered. The stock price at the end of all such paths will be $S(0)U^k D^{n-k}$, assuring the recombination of these paths, as we desired.

But what should be the value of U and D? We will choose these so that the expected return and variance on each time interval correspond to our earlier assumptions. Let us assume that we divide the time interval from 0 to T into n equal intervals of length Δ.

The i^{th} interval will span the time period $[i\Delta, (i+1)\Delta]$, $0 \leq i < n$.

We introduce the following definitions:

$$\begin{aligned} U &= \exp(\mu\Delta + \sigma\sqrt{\Delta}) \\ D &= \exp(\mu\Delta - \sigma\sqrt{\Delta}) \end{aligned}$$

In addition, we will assume that $p = \frac{1}{2}$.

We immediately get that the expectation of the stock price return on interval i is

$$\begin{aligned} E[r(i\Delta)] &= \mu\Delta \\ Var[r(i\Delta)] &= \sigma^2\Delta \end{aligned}$$

Thus the return on the stock price satisfies the assumptions that we introduced in the previous lecture on each interval of length Δ. We note here that our choice for U and D - as well as for p - is not unique. Other choices are possible.

Let us know consider paths that have k "up" states and $n-k$ "down" states. All these paths will end in the same state s, in which the price of the stock will be $S(T) = S(0)U^kD^{n-k}$, and the n-interval return on the stock price will be $r_T = k\log U + (n-k)\log D$. Using our values for U and D, we get:

$$\begin{aligned} r_T &= \mu k\Delta + \sigma k\sqrt{\Delta} + \mu(n-k)\Delta - \sigma(n-k)\sqrt{\Delta} \\ &= \mu n\Delta + \sigma(2k-n)\sqrt{\Delta} \\ &= \mu T + \sigma(2k-n)\sqrt{\frac{T}{n}} \\ S(T) &= S(0)\exp\left[\mu T + \sigma(2k-n)\sqrt{\frac{T}{n}}\right] \end{aligned}$$

Let us introduce a binomial variable X_n, whose value is equal to the number of "up" states that a certain stock price evolution "encounters" from time 0 to time T. We then get:

$$\begin{aligned} r_T &= \mu T + \sigma(2X_n - n)\sqrt{\frac{T}{n}} \\ S(T) &= S(0)\exp\left[\mu T + \sigma(2X_n - n)\sqrt{\frac{T}{n}}\right] \end{aligned}$$

We can easily compute the expectation and the variance of X_n:

$$\begin{aligned} E[X_n] &= \frac{1}{2}n \\ Var[X_n] &= \frac{1}{4}n \end{aligned}$$

We rewrite the expression for the return to emphasize the return and variance of X_n:

$$r_T = \mu T + \sigma\sqrt{T}\frac{X_n - \frac{n}{2}}{\sqrt{\frac{n}{4}}}$$

By the Central Limit Theorem, the quantity $\frac{X_n - \frac{n}{2}}{\sqrt{\frac{n}{4}}}$ tends to $N(0,1)$. In the limit, we thus get that stock price returns on the interval $[0,T]$ are normally distributed, and that

$$\begin{aligned} E[r_T] &= \mu T \\ Var[r_T] &= \sigma^2 T \end{aligned}$$

Thus the properties that we assumed for the one-period evolution of the stock price induce the required properties on the stock price return over the period $[0,T]$. Since the return of the stock price return is normal, the evolution of the stock price is log-normal, as we wanted it to be.

0.3 Complications

During our discussion of the evolution of stock prices we have disregarded the effect of dividends. When paid, say, at time t_d, such dividends induce a discontinuity in the stock price, since $S(t_d^-) = S(t_d^+) + dividend$. How does this impact the binomial model?

Let us consider two states, s_1 (with stock price S_1) and s_2 (with stock price S_2), in the lattice that could have been reached in one step (one time interval) from the same previous state s_* (with stock price S_*). If we assume that $S_1 > S_2$, we have that $S_1 = S_* U$, and $S_2 = S_* D$. If states s_1 and s_2 correspond to the payment of a dividend d, then the after-dividend prices of the stock will be $S_1^+ = S_* U - d$, and $S_2^+ = S_* D - d$. In the absence of dividends, the "down" state following s_1, and the "up" state following s_2 would recombine. Could this also happen in the presence of dividends? If yes, the following equalities must hold:

$$\begin{aligned} S_1^+ D &= S_2^+ U \\ (S_* U - d)D &= (S_* D - d)U \\ dD &= dU \end{aligned}$$

It is clear, however, that these equalities can only hold if $d = 0$, i.e. if there is no dividend paid in states s_1 and s_2.[3] Thus when discrete dividends are paid, the nodes of the lattice will not recombine.

It turns out that introducing discrete dividend payments can significantly increase the resource requirements imposed by a lattice computation. If we have a plain lattice with n time intervals (i.e. n edges between the root and a leaf node), then the total number of nodes in the lattice is equal to $\frac{(n+1)(n+2)}{2}$. How does this change if you have

[3] Or if $\sigma = 0$. This case would imply that there is no uncertainty w.r.t. the final values of the stock prices, as we would have $U = D$. We have excluded this possibility by assumption.

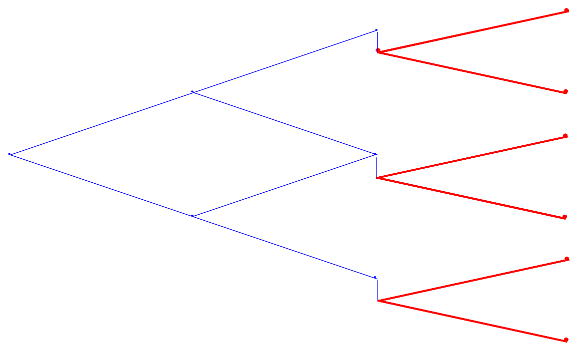

Figure 4: Lattice nodes will not recombine after a discrete dividend payment.

k dividend payments? Compute the total number of nodes in a lattice that has $n = km$ time intervals, and in which a discrete dividend is paid once every m intervals.

Because discrete dividends pose a computational problem, it is often assumed that dividends are paid continuously.